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Equilibria in a multi-period economy

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Abstract

We consider the model of a financial exchange economy with finitely many periods having financial restricted participation i.e., each agents portfolio choice is restricted to a closed convex set containing zero, as in Siconolfi [1989]. Time and uncertainty are represented by a finite event-tree. There is a market for physical commodities at any state today or in any date of future and financial transfers across time and across states are allowed by means of finitely many financial assets (nominal and numéraire assets). We prove a general existence result of equilibria for such a financial exchange economy in which agents may have non-ordered preferences.

Keywords: Multi-period economy, Restricted participation, Financial exchange economy, Arbitrage free prices, Equilibrium.

JEL Classification: C62, D52, D53.

1 Introduction

The main purpose of general equilibrium theory with incomplete markets is to study the interactions between the financial structure of the economy and the commodity structure, in a world in which time and uncertainty plays a fundamental role. The first pioneering multi-period model is due to Debreu [1959], who introduced the idea of an event-tree of finite length, in order to represent time and uncertainty in a stochastic economy. Later, Magill and Shafer [1991] extended the analysis of multi-period models, describing economies in which financial equilibria coincide with contingent market equilibria. The multi-period model was also explored, among others, by Duffie and Shafer [1985], who proved a result of generic existence of equilibria, a detailed presentation of which is provided in Magill and Quinzii [1996b].

The multi-period model has been also extensively studied in the simple 2-date model (one period $T = 1$): see, among others, Bich and Cornet [1997], Colell et al. [1995], Cass et al. [2001], for the case of a finite set of states and Colell and Zame [1996], Monteiro [1996], Araujo et al. [1997], Orrillo [2001] for the case of a continuum of states. The 2-date model, however, is not sufficient to capture the time evolution of realistic models. In this sense, the multi-period model is much more flexible, and is also a necessary intermediate step before studying the infinite horizon setting (see Magill and Quinzii [1994, 1996a]). Moreover, multi-period models may provide a framework for phenomena which do not occur in a simple 2-date model. For instance, Bonnisseau and Lachiri [2004] describes a 3-date economy with production in which, essentially, the second welfare theorem does not hold, while it always holds in the 2-date case. As a further example, we may recall that the suitable setting to study the effect of incompleteness of markets on price volatility is a 3-date model, in the way addressed in Citanna and Schmedders [2005].

In our model, we consider, time and uncertainty are represented by an event-tree with T periods and finitely many nodes (date-events) at each date. At each node, there is a spot market where a finite set of commodities are available. Moreover, transfers of value among nodes and dates are made possible via a financial structure, namely finitely many financial assets available at each node of the event-tree. Our equilibrium notion encompasses the case in which retrading of financial assets is allowed at every node (see Magill and Quinzii [1996b]) and we allow the case of restricted participation, namely the case in which agents portfolio sets may be constrained.

This paper focuses on the existence of financial equilibria in a stochastic economy with general financial assets. The existence problem with incomplete markets was studied, in the case of 2-date models, by Cass [1984] and Werner [1985, 1989] for nominal financial structures, Duffie [1987] for purely financial securities under general conditions, Geanakoplos and Polemarchakis [1986] in the case of numéraire assets. The existence of a financial equilibrium was proved by Bich and Cornet [1997], Aouani and Cornet [2009] when agents may have nontransitive preferences in the case of a 2-date economy. In the case of $T-$period economies, we also mention the work by Duffie and Shafer [1986] and by Florenzano and Gourdel [1994]; more recently, Martins-da-Rocha and Triki [2005] have studied a general intertemporal model in the case of financial securities. Other existence results in the infinite horizon models can be found in Levine and Zame [1996], Monteiro and Pascoa [2000], Florenzano et al. [2001].

2 The T-period financial exchange economy

2.1 Time and uncertainty in a multi-period model

We[1] consider a multi-period exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, \dots, T\}$, and a finite set of agents I. The stochastic structure of the model is described by a finite event-tree $\mathcal{D}$ of length T and we shall essentially use the same notations as in Magill and Quinzii [1996b] (we refer Magill and Quinzii [1996b] for an equivalent presentation with information partitions). The set $\mathcal{D}_t$ denotes the nodes (also called date-events) that could occur at date t and the family $(\mathcal{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathcal{D}$; we denote by $t(\xi)$ the unique $t \in \mathcal{T}$ such that $\xi \in \mathcal{D}_t$.

At each date $t \neq T$, there is an a priori uncertainty about which node will prevail in the next date. There is a unique non-stochastic event occurring at date $t = 0$, which is denoted ξ_0, (or simply 0) so $\mathcal{D}_0 = \{\xi_0\}$. Finally, the event-tree $\mathcal{D}$ is endowed with a predecessor mapping $pr : \mathcal{D} \setminus \{\xi_0\} \longrightarrow \mathcal{D}$ which satisfies $pr(\mathcal{D}_t) = \mathcal{D}_{t-1}$, for every $t \neq 0$. The element $pr(\xi)$ is called the immediate predecessor of ξ and is also denoted ξ^-. For each $\xi \in \mathcal{D}$, we let $\xi^+ = \{\bar{\xi} \in \mathcal{D} : \ \xi = \bar{\xi}^-\}$ be the set of immediate successors of ξ; we notice that the set ξ^+ is nonempty if and only if $\xi \in \mathcal{D} \setminus \mathcal{D}_T$.

Moreover, for $\tau \in \mathcal{T}\setminus\{0\}$ and $\xi \in \mathcal{D}\setminus\cup_{t=0}^{\tau-1}\mathcal{D}_t$ we define, by induction, $pr^\tau(\xi) = pr(pr^{\tau-1}(\xi))$ and we let the set of (not necessarily immediate) successors and the set of predecessors of ξ be respectively defined by

$$\mathcal{D}^+(\xi) = \{\xi' \in \mathcal{D} : \ \exists\tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^\tau(\xi)\},$$

$$\mathcal{D}^-(\xi) = \{\xi' \in \mathcal{D} : \ \exists\tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^\tau(\xi')\}.$$

If $\xi' \in \mathcal{D}^+(\xi)$ [resp. $\xi' \in \mathcal{D}^+(\xi) \cup \{\xi\}$], we shall also use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. We notice that $\mathcal{D}^+(\xi)$ is nonempty if and only if $\xi \notin \mathcal{D}_T$ and $\mathcal{D}^-(\xi)$ is nonempty if and only if $\xi \neq \xi_0$. Moreover, one has $\xi' \in \mathcal{D}^+(\xi)$ if and only if $\xi \in \mathcal{D}^-(\xi')$ (and similarly $\xi' \in \xi^+$ if and only if $\xi = (\xi')^-$).

[1] In this paper, we shall use the following notations. A $(\mathcal{D} \times J)-$matrix A is an element of $\mathbb{R}^{\mathcal{D}\times J}$, with entries $(a^j_\xi)_{\xi\in\mathcal{D},j\in J}$; we denote by $A_\xi \in \mathbb{R}^J$ the $\xi-$th row of A and by $A^j \in \mathbb{R}^{\mathcal{D}}$ the $j-$th column of A. We recall that the transpose of A is the unique $(J \times \mathcal{D})-$matrix A^t satisfying $(Ax) \bullet_{\mathcal{D}} y = x \bullet_J (A^t y)$, for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^{\mathcal{D}}$, where $\bullet_{\mathcal{D}}$ [resp. $\bullet_J$] denotes the usual scalar product in $\mathbb{R}^{\mathcal{D}}$ [resp. $\mathbb{R}^J$]. We shall denote by $rankA$ the rank of the matrix A. For every subsets $\widetilde{\mathcal{D}} \subset \mathcal{D}$ and $\widetilde{J} \subset J$, the $(\widetilde{\mathcal{D}} \times \widetilde{J})-$sub-matrix of A is the $(\widetilde{\mathcal{D}} \times \widetilde{J})-$matrix $\widetilde{A}$ with entries $\widetilde{a}^j_\xi = a^j_\xi$ for every $(\xi, j) \in \widetilde{\mathcal{D}} \times \widetilde{J}$. Let x, y be in $\mathbb{R}^n$; we shall use the notation $x \geq y$ (resp. $x \gg y$) if $x_h \geq y_h$ (resp. $x_h \gg y_h$) for every $h = 1, \cdots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : \ x \gg 0\}$. We shall also use the notation $x > y$ if $x \geq y$ and $x \neq y$. We shall denote by $\| \cdot \|$ the Euclidean norm in the different Euclidean spaces used in this paper and the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$ is denoted $B_L(x, r) := \{y \in \mathbb{R}^L : \ \|y - x\| \leq r\}$..

2.2 The stochastic exchange economy

At each node $\xi \in \mathcal{D}$, there is a spot market where a finite set H of divisible physical commodities is available. We assume that each commodity does not last for more than one period. In this model, a commodity is a couple (h, ξ) of a physical commodity $h \in H$ and a node $\xi \in \mathcal{D}$ at which it will be available, so the commodity space is $\mathbb{R}^L$, where $L = H \times \mathcal{D}$. An element x in $\mathbb{R}^L$ is called a *consumption*, that is $x = (x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$, where $x(\xi) = (x(h,\xi))_{h \in H} \in \mathbb{R}^H$, for every $\xi \in \mathcal{D}$.

We denote by $p = (p(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$ the vector of spot prices and $p(\xi) = (p(h,\xi))_{h \in H} \in \mathbb{R}^H$ is called the spot price at node ξ. The spot price $p(h,\xi)$ is the price paid at date $t(\xi)$, for the delivery of one unit of commodity h at node ξ. Thus the value of the consumption $x(\xi)$ at node $\xi \in \mathcal{D}$ (evaluated in unit of account of node ξ) is

$$p(\xi) \bullet_H x(\xi) = \sum_{h \in H} p(h,\xi) x(h,\xi).$$

There is a finite set I of consumers and each consumer $i \in I$ is endowed with a *consumption set* $X_i \subset \mathbb{R}^L$ which is the set of her possible consumptions. An *allocation* is an element $x \in \prod_{i \in I} X_i$, and we denote by x_i the consumption of agent i, that is the projection of x onto X_i.

We denote by $\mathbf{A}(\mathcal{E})$ the set of attainable allocations of the economy, that is

$$\mathbf{A}(\mathcal{E}) = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i \mid \sum_{i \in I} x_i = \sum_{i \in I} e_i\},$$

and by $\widehat{X}_i$ the projection of $\mathbf{A}(\mathcal{E})$ on X_i. Note that for every $i \in I$, $e_i \in \widehat{X}_i$.

The tastes of each consumer $i \in I$ are represented by a *strict preference correspondence* $P_i : \prod_{j \in I} X_j \longrightarrow X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred by i to x_i, that is, given the consumptions x^j for the other consumers $j \neq i$. Thus P_i represents the tastes of consumer i but also her behavior under time and uncertainty, in particular her impatience and her attitude towards risk. If consumers preferences are represented by utility functions $u_i : X_i \longrightarrow \mathbb{R}$, for every $i \in I$, the strict preference correspondence is defined by $P_i(x) = \{\bar{x}_i \in X_i \mid u_i(\bar{x}_i) > u_i(x_i)\}$.

Finally, at each node $\xi \in \mathcal{D}$, every consumer $i \in I$ has a *node-endowment* $e_i(\xi) \in \mathbb{R}^H$ (contingent to the fact that ξ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$ her *endowment vector* across the different nodes. The exchange economy $\mathcal{E}$ can thus be summarized by

$$\mathcal{E} = [\mathcal{D}; H; I; (X_i, P_i, e_i)_{i \in I}].$$

2.3 The financial structure

We consider finitely many financial assets and we denote by J the set of assets. An asset $j \in J$ is a contract, which is issued at a given and unique node in $\mathcal{D}$, denoted by ξ^j and called the *emission node* of j. $J(\xi)$ represents the set of assets with emission node ξ. Each asset j is bought (or sold) at its emission node ξ^j and only yields payoffs at the successor nodes ξ' of ξ^j, that is, for $\xi' > \xi^j$. To allow for real assets, we let the payoff depend upon the spot price vector $p \in \mathbb{R}^L$ and we denote by $V^j_{\xi'}(p)$ the payoff of asset j at node ξ'. For the sake of convenient notations, we shall in fact consider the payoff of asset j at every node $\xi' \in \mathcal{D}$ and assume that it is zero if ξ' is not a successor of the emission node ξ^j. Formally, we assume that $V^j_{\xi'}(p) = 0$ if $\xi' \notin \mathcal{D}^+(\xi^j)$. With the above convention, we notice that every asset has a zero payoff at the initial node, that is $V^j_0(p) = 0$ for every $j \in J$; furthermore, every asset j which is emitted at the terminal date has a zero payoff, that is, if $\xi^j \in \mathcal{D}_T$, then $V^j_\xi(p) = 0$ for every $\xi \in \mathcal{D}$.

For every consumer $i \in I$, if $z^j_i > 0$ [resp. $z^j_i < 0$], then $|z^j_i|$ will denote the quantity of asset $j \in J$ bought [resp. sold] by agent i at the emission node ξ^j. The vector $z_i = (z^j_i)_{j \in J} \in \mathbb{R}^J$ is called the *portfolio* of agent i.

We assume that each consumer $i \in I$, at every node $\xi \in \mathcal{D}$ is endowed with a *portfolio set* $Z_i(\xi) \subset \mathbb{R}^{J(\xi)}$, which represents the set of portfolios that are admissible for agent i at node ξ. We define $Z_i = \prod_{\xi \in \mathcal{D}} Z_i(\xi) \in \mathbb{R}^J$ as the portfolio set of agent i. This general framework allows us to treat, for example, the following important cases:

- $Z_i = \mathbb{R}^J$ (unconstrained portfolios);
- $Z_i \subset \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales);
- $Z_i = \prod_\xi B_{J(\xi)}(0, 1)$ (bounded portfolios).

The price of asset j is denoted by q^j and we recall that it is paid at its emission node ξ^j. We let $q = (q^j)_{j \in J} \in \mathbb{R}^J$ be the asset price (vector).

In multi-period economy, we can classify financial assets broadly in two categories:

i)Short Lived Asset :- An asset j is said to be short lived if it has non-zero payoffs only at the immediate successors of the node at which it is issued, *i*.e., $V^j_\xi(p) = 0$ for all $\xi \notin (\xi^j)^+$.

ii) Long Lived Asset :- An asset j is said to be short lived if it is not short lived.

Definition 1. *A* financial structure $\mathcal{F} = (J, (Z_i)_{i \in I}, (\xi^j)_{j \in J}, V)$ *consists of*

- *a set of assets J,*
- *a collection of portfolio sets $Z_i \subset \mathbb{R}^J$ for every agent $i \in I$,*
- *a node of issue $\xi^j \in \mathcal{D}$ for each asset $j \in J$,*
- *a payoff mapping $V : \mathbb{R}^L \to (\mathbb{R}^{\mathcal{D}})^J$ which associates, to every spot price $p \in \mathbb{R}^L$ the $(\mathcal{D} \times J)-$payoff matrix $V^j(p) = (V^j_\xi(p)))_{\xi \in \mathcal{D}}$, and satisfies the condition $V^j_{\xi'}(p) = 0$ if $\xi' \notin \mathcal{D}^+(\xi(j))$.*

The full matrix of payoffs $W_{\mathcal{F}}(p, q)$ *is the $(\mathcal{D} \times J)-$matrix with entries*

$$(W_{\mathcal{F}})^j_\xi(p, q) := \sum_{\xi^j \in \mathcal{D}^-(\xi)} V^j_\xi(p) - \delta_{\xi, \xi^j} q_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, for a given portfolio $z \in \mathbb{R}^J$ (and given prices (p, q)) the full flow of returns is $W_{\mathcal{F}}(p, q)z$ and the (full) financial return at node ξ is

$$[W_{\mathcal{F}}(p,q)z]_\xi := W_{\mathcal{F}}(p,q)_\xi \bullet_J z = \sum_{j \in J} V_\xi^j(p) z^j - \sum_{j \in J} \delta_{\xi, \xi^j} q^j z^j \tag{2.1}$$

$$= \sum_{\{j \in J | \ \xi^j < \xi\}} V_\xi^j(p) z^j - \sum_{\{j \in J | \ \xi^j = \xi\}} q^j z^j, \tag{2.2}$$

and we shall extensively use the fact that, for $\mu \in \mathbb{R}^{\mathcal{D}}$, and $j \in J$, one has:

$$[W_{\mathcal{F}}^t(p,q)\mu]^j = \sum_{\xi \in \mathcal{D}} \mu(\xi) V_\xi^j(p) - \sum_{\xi \in \mathcal{D}} \mu(\xi) \delta_{\xi, \xi^j} q^j \tag{2.3}$$

$$= \sum_{\xi > \xi^j} \mu(\xi) V_\xi^j(p) - \mu(\xi^j) q^j. \tag{2.4}$$

In the following, when the financial structure $\mathcal{F}$ remains fixed, while only prices vary, we shall simply denote by $W(p, q)$ the full matrix of returns. In the case of unconstrained portfolios, namely $Z_i = \mathbb{R}^J$, for every $i \in I$, the financial asset structure will be simply denoted by $\mathcal{F} = (Z, W)$ 0r equivalently $\mathcal{F} = (J, (Z_i)_{i \in I}, (\xi^j)_{j \in J}, V)$.

2.4 Financial equilibria

2.4.1 Financial equilibria without retrading

We now consider a financial exchange economy, which is defined as the couple of an exchange economy $\mathcal{E}$ and a financial structure $\mathcal{F}$. It can thus be summarized by

$$(\mathcal{E}, \mathcal{F}) := [\mathcal{D}, H, I, (X_i, P_i, e_i)_{i \in I}; J, (Z_i)_{i \in I}, (\xi^j)_{j \in J}, V].$$

Given the price $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, the *budget set* of consumer $i \in I$ is[2]

$$B_{\mathcal{F}}^i(p,q) = \{(x_i, z_i) \in X_i \times Z_i : \ \forall \xi \in \mathcal{D}, \ \ p(\xi) \bullet_H [x_i(\xi) - e_i(\xi)] \leq [W_{\mathcal{F}}(p,q) z_i]_\xi\} \tag{2.5}$$

$$= \{(x_i, z_i) \in X_i \times Z_i : \ p \,\square\, (x_i - e_i) \leq W_{\mathcal{F}}(p,q) z_i\}. \tag{2.6}$$

We now introduce the equilibrium notion.

Definition 2. *An* equilibrium *of the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a list of strategies and prices* $\left(\bar{x}, \bar{z}, \bar{p}, \bar{q}\right) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \setminus \{0\} \times \mathbb{R}^J$ *such that*
(a) for every $i \in I$, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preferences* P_i *in the budget set* $B_i(\bar{p}, \bar{q})$, *in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \ \textit{and} \ [P_i(\bar{x}) \times Z_i] \cap B_i(\bar{p}, \bar{q}) = \emptyset;$$

(b) $\displaystyle\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ *and* $\displaystyle\sum_{i \in I} \bar{z}_i = 0.$

[2]For $x = (x(\xi))_{\xi \in \mathcal{D}}, p = (p(\xi))_{\xi \in \mathcal{D}}$ in $\mathbb{R}^L = \mathbb{R}^{H \times \mathcal{D}}$ (with $x(\xi), p(\xi)$ in $\mathbb{R}^H$) we let $p \,\square\, x = (p(\xi) \bullet_H x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathcal{D}}$.

Proposition 1. *Assume the portofio sets Z_i are convex for every i. Under* **LNS***, if $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then $\bar{q}$ is (asymptotic-)arbitrage-free at $\bar{p}$, in the sense that*

$$W(\bar{p}, \bar{q})(\cup_i \mathbf{A} Z_i) \cap \mathbb{R}^{\bar{S}}_{+} = \{0\}.$$

We denote by $Q(p)$ the set of arbitrage-free prices at $p \in \mathbb{R}^L$.

Similarly, we define $Q_\xi(p)$ as set of arbitrage-free prices for the assets bought at node ξ, i.e., $\{j \mid \xi^j = \xi\}$ for $p \in \mathbb{R}^L$.

2.5 Assumptions on the model

In this section, we will discuss the assumptions of our model.

Consumption Assumption C

Now, we introduce the standard assumption on the consumption or exchange economy $\mathcal{E}$. For every $i \in I$

(i) X_i is a bounded below, closed, convex subset of $\mathbb{R}^L$.

(ii) **Continuity of Preferences** The correspondence $P_i : \prod_i X_i \to X_i$ is lower semicontinuous[3] with convex open values in X_i for the relative topology of X_i.

(iii) **Convexity of Preferences** $P_i(x)$ is convex for every x.

(iv) **Irreflexive Preferences** For every $x = (x_i)_{i \in I} \in \prod_i X_i$, $x_i \notin P_i(x)$.

(v) **Local Non-Satiation LNS**

(a) $\forall x \in \mathbf{A}(\mathcal{E})$, $\forall \xi \in \mathcal{D}$, $\exists x'_i \in P_i(x)$ such that $x'_i(\xi') = x_i(\xi')$ for $\xi' \neq \xi$.

(b) $[y_i \in P_i(x)]$ implies $(x_i, y_i] \subset P_i(x)$.

(vi) **Consumption Survival CS** For every $i \in I$, $e_i \in \text{int} X_i$.

Financial assumption F

Before defining the financial assumprtions, we introduce the notion $V(\xi, p)$ as the submatrix of $V(p)$, where we consider only the k-th column of $V(p)$, where $\xi^k = \xi$.

[3] a correspondence $\varphi : X \to Y$ is said to be lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \varphi(x_0)$ is non empty, there exists a neighborhood U of x_0 in X such that, for all $x \in U$, $V \cap \varphi(x)$ is nonempty. The correspondence φ is said to be lower semicontinuous if it is lower semicontinuous at each point of X.

Now we define standard assumptions on the financial structure $\mathcal{F}$.

F0 The set $\mathbf{A}_\xi(\mathcal{F}) := \sum_{i\in I}(\mathbf{A}Z_i(\xi) \cap \{V(\xi,p) \geq 0\})$ does not depend on p;

F1 For every $i \in I$, Z_i is closed, convex and $0 \in Z_i$, and the mapping $V(\xi,\cdot) : \mathbb{R}^L \to \mathbb{R}^{S\times J}$ is continuous $\forall \xi \in \mathcal{D}$;

F2 For every commodity price vector $p \in \mathbb{R}^L$, the sets $\mathbf{A}Z_i(\xi) \cap \ker V(\xi,p)$ are strongly positively semi-independent (**SPSI**), that is

$$\forall p \in \mathbb{R}^L, \Big(\sum_{i\in I}\mathbf{A}Z_i(\xi) \cap \ker V(\xi,p)\Big) \cap -\Big(\sum_{i\in I}\mathbf{A}Z_i(\xi) \cap \ker V(\xi,p)\Big) = \{0\};$$

F3 Financial Survival For every $i \in I$, for every $p \in \mathbb{R}^L$ such that $p(\xi) = 0$ for some $\xi \in \mathcal{D} \setminus \mathcal{D}_T$, and for every $q(\xi) \in \mathrm{cl}Q_\xi(p) \cap Z_{\mathcal{F}}(\xi)$, $q(\xi) \neq 0$, there exists $z_i(\xi) \in Z_i(\xi)$ such that $q(\xi) \cdot z_i(\xi) < 0$;

F4 Arbitrage-free The set $Q(p)$ is a convex cone for every $p \in \mathbb{R}^L$.

Theorem 1. *Let $(\mathcal{E},\mathcal{F})$ be a financial exchange economy satisfying assumptions* **C**, *and* **F**. *Then it admits an equilibrium $(\bar{p},\bar{q},\bar{x},\bar{z})$ with $\bar{q} \in$ closure of $Q(\bar{p})$ such that $||\bar{p}(\xi)|| + ||\bar{q}(\xi)|| = 1$,$\forall \xi \in \mathcal{D} \setminus \mathcal{D}_T$ and $||\bar{p}(\xi)|| = 1$ for $\xi \in \mathcal{D}_T$.*

The proof of Theorem 1 is done in next Section 3.

2.6 A corollary of the existence result

We take the definition of the equivalent exchange economy as defined in a companion paper (see Cornet and Ranjan [2012b]).

Definition 3. *The two financial structures $\mathcal{F}$ and $\mathcal{F}'$ are said to be equivalent, denoted $\mathcal{F} \sim \mathcal{F}'$, if for every standard exchange economy $\mathcal{E}$, the financial exchange economies $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E},\mathcal{F}')$ have the same consumption equilibria.*

Now we take an assumption about the existence of an equivalent exchange economy with the convex cone set of arbitrage-free prices.

F4' There exist an equivalent financial structure $\mathcal{F}'$, where the set $Q_{\mathcal{F}'}(p)$ is a convex cone for every $p \in \mathbb{R}^L$.

Now the following corollary of the Theorem 1 is straightforward.

Corollary 1. *Let $(\mathcal{E},\mathcal{F})$ be a financial exchange economy satisfying assumptions* **C, F0, F1, F2, F3** *and* **F4'**. *Then it admits an equilibrium $(\bar{p},\bar{q},\bar{x},\bar{z})$.*

3 Proof of Theorem 1

3.1 Preliminary results

We first state the following lemmas that will be used hereafter. The proof of these lemmas are done in Appendix.

Lemma 1. *For every $p \in \mathbb{R}^L$, the set $Q_\xi(p)$ is a convex cone with vertex 0.*

Lemma 2. *If $Q(p)$ is a convex cone, then the set $Q(p) = \prod_{\xi \in \mathcal{D}} Q_\xi(p)$.*

We define $\Pi = \{(p,q) \in \mathbb{R}^L \times \mathbb{R}^J \mid \forall \xi \in \mathcal{D}_T, ||p(\xi)|| \leq 1,\ and\ \forall \xi \in \mathcal{D} \setminus \mathcal{D}_T,\ q(\xi) \in Q_\xi \cap Z_{\mathcal{F}}(\xi)\ and\ ||p(\xi)|| + ||q(\xi)|| \leq 1\}$.

Lemma 3. *The set Π is convex, compact, and $(0,0) \in \Pi$.*

Lemma 4. *Under assumption $F0$ and $F2$, $\forall v = (v_i)_i \in (\mathbb{R}^{\mathcal{D}\setminus\{0\}})^I$ the set K_v is bounded for a given $q \in \mathrm{cl}Q(p)$, where*

$$K_v := \{(z_1, ..., z_I, p) \in (\prod_i Z_i) \times B_L(0,1) : \sum_{j | \xi^j \in \mathcal{D}^-(\xi)} V_\xi^j(p) z_i^j - q(\xi) \cdot z_i(\xi) \geq v_i^\xi,$$

$$\forall \xi \in \mathcal{D} \setminus (\mathcal{D}_T \cup 0)\ and \sum_{j | \xi^j \in \mathcal{D}^-(\xi)} V_\xi^j(p) z_i^j \geq v_i^\xi,\ \forall \xi \in \mathcal{D}_T\ and$$

$$-\sum_{i \in I} z_i(\xi) \in \sum_{i \in I} (\mathbf{A} Z_i(\xi) \cap \{V(\xi,p) \geq 0\}) \forall \xi \in \mathcal{D} \setminus \mathcal{D}_T\}.$$

3.2 Truncating the economy

It follows from the Assumption **C(i)** that the set $\mathbf{A}(\mathcal{E})$ is bounded. We denote by $\hat{X}_i$ the projection of $\mathbf{A}(\mathcal{E})$ on X_i and that for every $i \in I$, $e_i \in \hat{X}_i$. Hence the set $\hat{X}_i$ is bounded, for every $i \in I$. Consequently, one can choose $r_1 > 0$ large enough such that

$$\hat{X}_i \subset int B_L(0, r_1) \text{ for every } i \in I.$$

For $i \in I$, let $\underline{v}_i \in \mathcal{R}^{\mathcal{D}}$ be defined by: for every $\xi \in \mathcal{D} \setminus \{0\}$,

$$\underline{v}_i^\xi = -1 + min\Big\{p(\xi).(x_i(\xi) - e_i(\xi)) - 1, p \in B_l(0,1), x_i \in B_L(0, r_1)\Big\} \tag{3.1}$$

The existence of $\underline{v}_i^\xi$ follows from the compactness of $B_l(0,1)$ and $B_L(0,r_1)$. We denote by $\hat{Z}_i$ the projection of Z_i on $K_{\underline{v}_i^\xi}$, $\forall \xi \in (\xi^-)^+$, and hence $\hat{Z}_i$ is bounded for every $i \in I$. Consequently, one can choose $r_2 > 0$ large enough such that

$$\hat{Z}_i \subset int B_J(0, r_2) \text{ for every } i \in I.$$

We let for every $i \in I$,

$$\begin{aligned} X_i^r &= X_i \cap \mathrm{int} B_L(0,r) \\ P_i^r(x) &= P_i(x) \cap \mathrm{int} B_L(0,r), \text{ and} \\ Z_i^r &= Z_i \cap \mathrm{int} B_J(0,r), \end{aligned}$$

and we define a new financial economy $(\mathcal{E}^r, \mathcal{F}^r)$ where the consumption sets are X_i^r, the preference correspondences are P_i^r, and the portfolio sets are Z_i^r. To summarize, we let

$$(\mathcal{E}^r, \mathcal{F}^r) := \Big((X_i^r, P_i^r, e_i)_{i \in I}, (W, (Z_i^r)_{i \in I}) \Big)$$

Note that, for every $i \in I$, $e_i \in \hat{X}_i$.

3.3 Definition of the reaction correspondences

Given $(p, q) \in \Pi$, following ideas originating from the Bergstrom [1976], we define the "modified" budget set of consumer i as follows:

$$B_i^{r\varepsilon}(p,q) = \{(x_i, z_i) \in X_i^r \times Z_i^r \mid p \,\square\, (x_i - e_i) \le W(p,q) z_i + \varepsilon(p,q)\};$$

$$\breve{B}_i^{r\varepsilon}(p,q) = \{(x_i, z_i) \in X_i^r \times Z_i^r \mid p \,\square\, (x_i - e_i) << W(p,q) z_i + \varepsilon(p,q)\},$$

where $\varepsilon(p,q) = (\varepsilon_\xi(p,q))_{\xi \in \mathcal{D}} \in \mathbb{R}^{\mathcal{D}}$ is defined by

$$\varepsilon_\xi(p,q) = \begin{cases} 1 - ||p(\xi)|| - ||q(\xi)|| & \text{if } \xi \in \mathcal{D} \setminus \mathcal{D}_T, \\ 1 - ||p(\xi)|| & \text{if } \xi \in \mathcal{D}_T. \end{cases} \tag{3.2}$$

Claim 3.1. *For all $(p,q) \in \Pi, \breve{B}_i^{r\varepsilon}(p,q) \neq \phi$ and moreover $B_i^{r\varepsilon}(p,q) = cl \breve{B}_i^{r\varepsilon}(p,q)$.*

Proof : Let $(p,q) \in \Pi$. Since $e_i \in \mathrm{int} X_i$, $\exists x_i \in X_i^r$ such that $p \,\square\, (x_i - e_i) \le 0$ with a strict inequality at each $\xi \in \mathcal{D}$ such that $p(\xi) \neq 0$. If $p(\xi) \neq 0$ for all $\xi \in \mathcal{D} \setminus \mathcal{D}_T$, then clearly $(x_i, 0) \in \breve{B}_i^{r\varepsilon}(p,q)$. Also, if for some $\xi \in \mathcal{D} \setminus \mathcal{D}_T$, $p(\xi) = 0$ and $q(\xi) = 0$, then $(x_i, 0) \in \breve{B}_i^{r\varepsilon}(p,q)$.

Now assume for some $\xi \in \mathcal{D} \setminus \mathcal{D}_T$, $p(\xi) = 0$ and $q(\xi) \neq 0$. For all $\xi \in \mathcal{D} \setminus \mathcal{D}_T$ such that $p(\xi) = 0$ and $q(\xi) \neq 0$, there exist $z_i(\xi) \in Z_i(\xi)$ such that $q(\xi) \cdot z_i(\xi) < 0$. Now, we want to find z'_i such that the following equation is satisfied for all $\xi \in \mathcal{D}$;

$$p(\xi) \cdot (x_i(\xi) - e_i(\xi)) + q(\xi) \cdot z'_i(\xi) - \sum_{j \mid \xi^j \in \mathcal{D}^-(\xi)} V_\xi^j(p) z_i'^j - \varepsilon_\xi(p,q) < 0.$$

This equation is equivalent to

$$p(\xi)\cdot(x_i(\xi)-e_i(\xi))+q(\xi)\cdot z'_i(\xi)-\sum_{j|\xi^j\in\mathcal{D}^-(\xi)}V^j_\xi(p)z'^j_i-\varepsilon_\xi(p,q)<0. \tag{3.3}$$

Now for all $\xi\in\mathcal{D}$ such that $p(\xi)\neq 0$ or $q(\xi)=0$, we take $z_i(\xi')=0$ and for all $\xi\in\mathcal{D}\setminus\mathcal{D}_T$, we take $z'_i(\xi')=t(\xi')z_i(\xi')$. Equation 3.3 implies

$$p(\xi)\cdot(x_i(\xi)-e_i(\xi))+t(\xi)q(\xi)\cdot z_i(\xi)-\sum_{j|\xi'^j\in\mathcal{D}^-(\xi)}t(\xi'^j)V^j_\xi(p)z^j_i-\varepsilon_\xi(p,q)<0. \tag{3.4}$$

We can find $t(0)>0$ and then $t(\xi)>0$ for $\xi\in 0^+$, and so on $t(\xi)>0$ for all $\xi\in\mathcal{D}_T$ such that Equation 3.4 is satisfied. Therefore $\breve{B}^{r\varepsilon}_i(p,q)\neq\phi$, and $B^{r\varepsilon}_i(p,q)=\mathrm{cl}\breve{B}^{r\varepsilon}_i(p,q)$ is obvious. □

Claim 3.2. *For all $i\in I$, $B^{r\varepsilon}_i$ is lower semicontinuous and upper semicontinuous on Π with closed convex values.*

Proof : From Claim 3.1, $B^{r\varepsilon}_i(p,q)$ is the closure of $\breve{B}^{r\varepsilon}_i(p,q)$ on Π. $\breve{B}^{r\varepsilon}_i(p,q)$ is lower semicontinuous, since it is an open graph. Therefore, $B^{r\varepsilon}_i(p,q)$ closure of lower semicontinuous correspondence and hence lower semicontinuous. Furthermore, $B^{r\varepsilon}_i(p,q)$ has a closed graph with convex values in the compact convex set $(X^r_i\times Z^r_i)$, and hence upper semicontinuous. □

Following the ideas from Gale and Mas-Colell [1975], we define a new function φ in following way. For $i\in I$,

$$\varphi_i(p,q,x,z)=\begin{cases}B^{r\varepsilon}_i(p,q) & \text{if } (x_i,z_i)\notin B^{r\varepsilon}_i(p,q),\\ \breve{B}^{r\varepsilon}_i(p,q)\cap(P^r_i(x)\times Z^r_i) & \text{otherwise,}\end{cases}$$

for $i=0$,

$$\varphi_0(p,q,x,z)=\Big\{(p',q')\in\Pi\mid(p'-p).\textstyle\sum_{i\in I}(x_i-e_i)+(q'-q).\sum_{i\in I}z_i>0\Big\}.$$

Remark 1. *By construction, for every $\xi\in\mathcal{D}\setminus\mathcal{D}_T$, $(p,q)\notin\varphi_0(p,q,x,z)$, and for every $i\in I$, whenever $(x_i,z_i)\notin B^{r\varepsilon}_i(p,q)$, then $\varphi_i(p,q,x,z)=B^{r\varepsilon}_i(p,q)$ and $(x_i,z_i)\notin\varphi_i(p,q,x,z)$.*

Claim 3.3. *For every $i\in\{0\}\cup I$, the correspondence φ_i is lower semicontinuous with convex values on $\Pi^n\times\prod_{i\in I}(X^r_i\times Z^r_i)$.*

Proof : When $i=0$, The correspondence φ_0 has an open graph thus it is lower semicontinuos and convexity follows trivially. And if $i\in I$, it follows from lower and upper semicontinuity of $B^{r\varepsilon}_i(p,q)$ that φ_i is lower semicontinuous at (p,q,x,z) if $(x_i,z_i)\notin B^{r\varepsilon}_i(p,q)$, since $\varphi_i=B^{r\varepsilon}_i(p,q)$ on a neighborhood of (x_i,z_i) which does not intersect the graph of $B^{r\varepsilon}_i(p,q)$. If $(x_i,z_i)\in B^{r\varepsilon}_i(p,q)$, then $\breve{B}^{r\varepsilon}_i(p,q)\cap(P^r_i(x)\times Z^r_i)$ is lower semicontinuous since $\breve{B}^{r\varepsilon}_i(p,q)$

has an open graph and $(P_i^r(x) \times Z_i^r)$ is lower semicontinuous. Thus φ_i is lower semicontinuous at (p,q,x,z) since $\breve{B}_i^{r\varepsilon}(p,q) \subset B_i^{r\varepsilon}(p,q)$ which clearly implies $\varphi_i(p,q,x,z) \subset B_i^{r\varepsilon}(p,q)$. The convexity of the values of φ_i is a consequence of the convexity of $\breve{B}_i^{r\varepsilon}(p,q)$, $B_i^{r\varepsilon}(p,q)$, Z_i^r and $P_i^r(x)$. $\square$

3.4 The fixed point argument

Theorem by Gale and Mas-Colell [1975]

Theorem 2. *Let $\mathcal{I}_0$ be a finite set, let $\mathcal{C}_i, i \in \mathcal{I}_0$ be a nonempty, compact, convex subset of some euclidean space, let $\mathcal{C} = \Pi_{i\in\mathcal{I}_0}\mathcal{C}_i$ and let ψ_i $(i \in \mathcal{I}_0)$ be a correspondence from $\mathcal{C}$ to $\mathcal{C}_i$, which is lower semicontinuous and convex-valued. Then, there exists $c^* = (c_i^*)_i \in \mathcal{C}$ such that, for every $i \in \mathcal{I}_0$ [either $c_i^* \in \psi(c_i^*)$ or $\psi(c_i^*) = \emptyset$].*

It follows from Theorem 2 that there exists

$$(\bar{p},\bar{q},\bar{x},\bar{z}) \in \Pi \times \prod_{i\in I}(X_i^r \times Z_i^r)$$

such that $\forall i \in I$, either $\varphi_i(\bar{p},\bar{q},\bar{x},\bar{z}) = \emptyset$ or $(\bar{x},\bar{z}) \in \varphi_i(\bar{p},\bar{q},\bar{x},\bar{z})$, and for $i = 0$, either $\varphi_0(\bar{p},\bar{q},\bar{x},\bar{z}) = \emptyset$ or $(\bar{p},\bar{q}) \in \varphi_0(\bar{p},\bar{q},\bar{x},\bar{z})$. And from Remark 1, we know that,

$$(\bar{p},\bar{q}) \notin \varphi_0(\bar{p},\bar{q},\bar{x},\bar{z}) \text{ and } (\bar{x},\bar{z}) \in B_I^{r\varepsilon}(\bar{p},\bar{q}). \tag{3.5}$$

Since $(\bar{p},\bar{q}) \notin \varphi_0(\bar{p},\bar{q},\bar{x},\bar{z})$, implies $\varphi_0(\bar{p},\bar{q},\bar{x},\bar{z}) = \emptyset$. and therefore,

$$p.\sum_{i\in I}(\bar{x}_i - e_i) + q.\sum_{i\in I}\bar{z}_i \le \bar{p}.\sum_{i\in I}(\bar{x}_i - e_i) + \bar{q}.\sum_{i\in I}\bar{z}_i,\ \forall (p,q) \in \Pi. \tag{3.6}$$

From 4.1, we deduce for every $(p,q) \in \Pi$ and for every $\xi \in \mathcal{D} \setminus \mathcal{D}_T$,

$$p(\xi).\sum_{i\in I}(\bar{x}_i(\xi) - e_i(\xi)) + q(\xi).\sum_{i\in I}\bar{z}_i(\xi) \le \bar{p}(\xi).\sum_{i\in I}(\bar{x}_i(\xi) - e_i(\xi)) + \bar{q}(\xi).\sum_{i\in I}\bar{z}_i(\xi). \tag{3.7}$$

3.5 Checking the portfolio and commodity market clearing conditions

Since the Market Clearing Condition $\sum_{i\in I}\bar{z}_i = 0$ may not be satisfied by the portfolios $\bar{z} = (\bar{z}_i)_i$, the purpose of the next claim is to define new portfolios $\bar{\bar{z}}_i \in Z_i^r$ $(i \in I)$ that will satisfy the Portfolio Market Clearing Condition $\sum_{i\in I}\bar{\bar{z}}_i = 0$.
We let $\bar{\bar{z}}_i(\xi) = \bar{z}_i(\xi) + \zeta_i(\xi)$, for some $\zeta_i(\xi) \in \mathbf{A}Z_i(\xi) \cap \{V(\xi,\bar{p}) \ge 0\}$ $(i \in I)$ such that $\sum_{i\in I}\bar{z}_i(\xi) = -\sum_{i\in I}\zeta_i(\xi)$.

Claim 3.4. *For every $i \in I$, $\sum_{i\in I}\bar{\bar{z}}_i = 0$, $\bar{\bar{z}}_i \in \widehat{Z}_i \subset Z_i^r$, and for every $\xi \in \mathcal{D} \setminus \mathcal{D}_T$, $\bar{q}(\xi) \cdot \bar{\bar{z}}_i(\xi) = \bar{q}(\xi) \cdot \bar{z}_i(\xi)$, $V(\xi,\bar{p})\bar{\bar{z}}_i(\xi) \ge V(\xi,\bar{p})\bar{z}_i(\xi)$ and $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$.*

Proof. Firstly, $\sum_{i\in I}\bar{\bar{z}}_i = 0$ is obvious from the definition of $\bar{\bar{z}}_i(\xi)$. Also, $V(\xi,\bar{p})\bar{\bar{z}}_i(\xi) \geq V(\xi,\bar{p})\bar{z}_i(\xi)$ follows from the definition of ζ. Second, we need to show that $\bar{\bar{z}}_i \in \widehat{Z}_i \subset Z_i^r$, and $\bar{q}(\xi)\cdot\bar{\bar{z}}_i(\xi) = \bar{q}(\xi)\cdot\bar{z}_i(\xi)$, or $\bar{q}(\xi)\cdot\zeta_i(\xi) = 0$.

We will prove these results by induction. We will show that the result holds for $\xi = 0$, and then will show that the result holds for ξ under the assumption that the result holds for all $\xi' \in \xi^-$. We claim $\sum_{i\in I}\bar{z}_i(0) \in Q_0^o$. Suppose our claim is not true then there exist $q' \in Q$ such that $q'(0)\cdot(\sum_{i\in I}\bar{z}_i(0)) > 0$. Without any loss of generality, we can assume that $q' \in B_J(0,1)$. From (4.5) we have

$$0 < q'(0)\cdot\sum_{i\in I}\bar{z}_i(0) \leq \bar{p}(0)\cdot\sum_{i\in I}(\bar{x}_i(0) - e_i(0)) + \bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0). \tag{3.8}$$

Since $(\bar{x}_i,\bar{z}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$ (by (3.5)) we deduce that

$$\bar{p}(0)\cdot\big(\bar{x}_i(0) - e_i(0)\big) + \bar{q}(0)\cdot\bar{z}_i(0) \leq \varepsilon_0(\bar{p},\bar{q}) \quad \text{for all} \quad i \in I.$$

Summing up over i we get

$$\bar{p}(0)\cdot\sum_{i\in I}\big(\bar{x}_i(0) - e_i(0)\big) + \bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0) \leq \varepsilon_0(\bar{p},\bar{q})I,$$

which together with the above inequality (3.8) implies that $\varepsilon_0(\bar{p},\bar{q}) > 0$.

We now claim that $\|\bar{p}(0)\| + \|\bar{q}(0)\| = 1$. Indeed, otherwise $\|\bar{p}(0)\| + \|\bar{q}(0)\| < 1$ and there exists $\alpha > 1$ such that $\|\alpha\bar{p}(0)\| + \|\alpha\bar{q}(0)\| < 1$ and $\alpha\bar{q}(0) \in \mathrm{cl}Q_0(\bar{p}) \cap Z_{\mathcal{F}}$ (since the latter set is a cone). Consequently, from (4.5), (taking $(p,q) \in \Pi$ defined by $p(0) = \alpha\bar{p}(0)$, $p(\xi) = \bar{p}(\xi)$ for $\xi \neq 0$, $q(0) = \alpha\bar{q}(0)$ and $q(\xi) = \bar{q}(\xi)$ for $\xi \neq 0$) we deduce that:

$$\alpha\bar{p}(0)\cdot\sum_{i\in I}\big(\bar{x}_i(0) - e_i(0)\big) + \alpha\bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0) \leq \bar{p}(0)\cdot\sum_{i\in I}\big(\bar{x}_i(0) - e_i(0)\big) + \bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0).$$

Dividing by $\bar{p}(0)\cdot\sum_{i\in I}\big(\bar{x}_i(0) - e_i(0)\big) + \bar{q}\cdot\sum_{i\in I}\bar{z}_i > 0$ (by inequality (3.8)), we get $\alpha \leq 1$, which contradicts that $\alpha > 1$.

Finally, we show that $\bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0) = 0$. We have $\bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0) \leq 0$ since $\bar{q}(0) \in Q_0$ and $\sum_{i\in I}\bar{z}_i(0) \in Q_0^o$ (from above). Taking (p,q) such that $p = \bar{p}, q(0) = 0$ and $q(\xi) = \bar{q}(\xi)$ in Π (in (4.5)), we deduce that $0 \leq \bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0)$. Hence, $\bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0) = 0$.

Now we let $\bar{z}_i'(0) = \bar{z}_i(0) + \zeta_i(0)$, for some $\zeta_i(0) \in \mathbf{A}Z_i(0) \cap \{V(0,\bar{p}) \geq 0\}$ $(i \in I)$ such that $\sum_{i\in I}\bar{z}_i(0) = -\sum_{i\in I}\zeta_i(0)$. Clearly $\sum_{i\in I}\bar{z}_i'(0) = 0$. We define z_i^1 such that $z_i^1(0) = z_i'(0)$ and $z_i^1(\xi) = \bar{z}_i(\xi)$ for $\xi \neq 0$.

We now show that $\bar{q}(0)\cdot(\bar{z}_i^1(0) - \bar{z}_i(0)) = \bar{q}(0)\cdot\zeta_i(0) = 0$ for every $i \in I$. We claim that $-\bar{q}(0)\cdot\zeta_i(0) \leq 0$ for every i. Let's say that $-\bar{q}(0)\cdot\zeta_i(0) > 0$ for some $i \in I$, and since $\zeta_i \in \mathbf{A}Z_i(0) \cap \{V(0,\bar{p}) \geq 0\}$, we have a contradiction to the fact that $\bar{q}(0) \in Q_0$.

Now recalling that $\bar{q}(0)\cdot\sum_{i\in I}\zeta_i(0) = -\bar{q}(0)\cdot\sum_{i\in I}\bar{z}_i(0) = 0$, we deduce that $\bar{q}(0)\cdot\zeta_i(0) = 0$ for every $i \in I$.

Therefore, $(\bar{x}_i, z_i^1) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$, and therefore $z_i^1 \in K_{\underline{v}_i^\xi}$ (from the definition of $\underline{v}_i^\xi$ in Equation 3.1).

Now we assume that these results hold for all $\xi' \in \mathcal{D}^-(\xi)$. We claim $\sum_{i\in I}\bar{z}_i(\xi) \in Q_\xi^o$. Suppose our claim is not true then there exist $q' \in Q$ such that $q'(\xi)\cdot(\sum_{i\in I}\bar{z}_i(\xi)) > 0$. Without any loss of generality, we can assume that $q' \in B_J(0,1)$. From (4.5) we have

$$0 < q'(\xi)\cdot\sum_{i\in I}\bar{z}_i(\xi) \le \bar{p}(\xi)\cdot\sum_{i\in I}(\bar{x}_i(\xi) - e_i(\xi)) + \bar{q}(\xi)\cdot\sum_{i\in I}\bar{z}_i(\xi). \tag{3.9}$$

Since $(\bar{x}_i, \bar{z}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$ (by (3.5)) we deduce that

$$\bar{p}(\xi)\cdot\big(\bar{x}_i(\xi) - e_i(\xi)\big) + \bar{q}(\xi)\cdot\bar{z}_i(\xi) \le \sum_{j|\xi^j\in\mathcal{D}^-(\xi)} V_\xi^j(p)z_i^j + \varepsilon_\xi(\bar{p},\bar{q}) \quad \text{for all} \quad i \in I.$$

Summing up over i we get

$$\bar{p}(\xi)\cdot\sum_{i\in I}\big(\bar{x}_i(\xi) - e_i(\xi)\big) + \bar{q}(\xi)\cdot\sum_{i\in I}\bar{z}_i(\xi) \le \sum_{i\in I}\sum_{j|\xi^j\in\mathcal{D}^-(\xi)} V_\xi^j(p)z_i^j + \varepsilon_\xi(\bar{p},\bar{q})I,$$

since $\sum_{i\in I}z_i^j = 0$ for all j such that $\xi^j \in \mathcal{D}^-(\xi)$ (from induction assumption)

$$\bar{p}(\xi)\cdot\sum_{i\in I}\big(\bar{x}_i(\xi) - e_i(\xi)\big) + \bar{q}(\xi)\cdot\sum_{i\in I}\bar{z}_i(\xi) \le 0 + \varepsilon_\xi(\bar{p},\bar{q})I,$$

which together with the above inequality (3.9) implies that $\varepsilon_\xi(\bar{p},\bar{q}) > 0$.

Now we claim $\|\bar{p}(\xi)\| + \|\bar{q}(\xi)\| = 1$. Proof of claim is similar to the proof in case $\xi = 0$, and we follow similar steps, as in case $\xi = 0$ to prove that $\bar{q}(\xi)\cdot\zeta_i(\xi) = 0$. And so we prove that $\bar{q}(\xi)\cdot\bar{\bar{z}}_i(\xi) = \bar{q}(\xi)\cdot\bar{z}_i(\xi)$ and $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$. Therefore $\bar{\bar{z}}_i \in K_{\underline{v}_i^\xi}$ (from the definition of $\underline{v}_i^\xi$ in Equation 3.1), which ends the proof. ∎

Now we will show the Market Clearing Condition for the commodity markets.

Claim 3.5. $\sum_{i\in I}\bar{x}_i = \sum_{i\in I}e_i$.

Proof. We first prove that the equality holds at node $\xi \in \mathcal{D}\setminus\mathcal{D}_T$. If $\sum_{i\in I}\bar{x}_i(\xi) \neq \sum_{i\in I}e_i(\xi)$, we deduce from (4.5), (taking $(p,q) \in \Pi$ defined by $p(\xi) = \sum_{i\in I}(\bar{x}_i(\xi) - e_i(xi))/\|\sum_{i\in I}(\bar{x}_i(\xi) - e_i(\xi))\|$, $p(\xi') = \bar{p}(\xi')$ for $\xi' \neq \xi$, $q(\xi) = 0$ and $q(\xi') = \bar{q}(\xi)$ for $\xi' \neq \xi$) that

$$0 < \|\sum_{i\in I}(\bar{x}_i(\xi) - e_i(\xi))\| \le \bar{p}(\xi)\cdot\sum_{i\in I}\big(\bar{x}_i(\xi) - e_i(\xi)\big) + \bar{q}(\xi)\cdot\sum_{i\in I}\bar{z}_i(\xi),$$

and in the exact same way as for inequality (3.8) in the proof of Claim 3.4 we obtain a contradiction. We now prove that the equality holds for all state $\xi \in \mathcal{D}_T$. Suppose that, for some $\xi \in \mathcal{D}_T$, $\sum_{i\in I}\bar{x}_i(\xi) \neq \sum_{i\in I} e_i(\xi)$. From (4.5), we deduce $\varepsilon_\xi(\bar{p},\bar{q}) = 0$, and

$$0 < \bar{p}(\xi) \cdot \sum_{i\in I}\big(\bar{x}_i(\xi) - e_i(\xi)\big).$$

Since $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$ (by Claim 3.4), and $\varepsilon_\xi(\bar{p},\bar{q}) = 0$, we have $\bar{p}(\xi) \cdot \big(\bar{x}_i(\xi) - e_i(\xi)\big) \leq \sum_{j|\xi^j \in \mathcal{D}^-(\xi)} V_\xi^j(\bar{p}) \cdot \bar{\bar{z}}_i^j$ for all $i \in I$. Summing up over i, and using the fact that $\sum_{i\in I}\bar{\bar{z}}_i = 0$ (by Claim 3.4) we get $\bar{p}(\xi) \cdot \sum_{i\in I}\big(\bar{x}_i(\xi) - e_i(\xi)\big) \leq 0$, a contradiction with the above strict inequality. ■

3.6 The point $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$

To show that the list $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$, we need to show that $\varepsilon(\bar{p},\bar{q}) = 0$, and for every $i \in I$,

$$B_i^{r\varepsilon}(\bar{p},\bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset. \tag{3.10}$$

Claim 3.6. *For each consumer $i \in I$, $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$, and $B_i^{r\varepsilon}(\bar{p},\bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$.*

Proof. From Claim 3.4, we know that $(\bar{x}_i, \bar{\bar{z}}_i)$ belongs to $B_i^{r\varepsilon}(\bar{p},\bar{q})$ for each $\imath \in I$.

We now prove that $B_i^{r\varepsilon}(\bar{p},\bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$. Since P_i^r has open values and since $B_i^{r\varepsilon}(\bar{p},\bar{q}) = \mathrm{cl}\, \breve{B}_i^{r\varepsilon}(\bar{p},\bar{q})$ (from Claim 3.1), implies that $B_i^{r\varepsilon}(\bar{p},\bar{q}) \cap (P_i^r(\bar{x}) \times Z_i^r) = \emptyset$. □

Claim 3.7. $\varepsilon(\bar{p},\bar{q}) = 0$*, that is, $||\bar{p}(\xi)|| + ||\bar{q}(\xi)|| = 1$, for all $\xi \in \mathcal{D} \setminus \mathcal{D}_T$ and $||\bar{p}(\xi)|| = 1$, for all $\xi \in \mathcal{D}_T$. Hence, $B_i^{r\varepsilon}(\bar{p},\bar{q}) = B_i^r(\bar{p},\bar{q})$.*

Proof. From Claim 3.6, we have $(\bar{x}_i, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$ for each $i \in I$, and we claim that the budget inequality is binding, that is:

$$\bar{p} \,\square\, (\bar{x}_i - e_i) = W(\bar{p},\bar{q})\bar{\bar{z}}_i + \varepsilon(\bar{p},\bar{q}) \qquad \forall i \in I \tag{3.11}$$

Indeed, if it is not true then there exists $\xi \in \mathcal{D}$ such that $\bar{p}(\xi) \cdot (\bar{x}_i(\xi) - e_i(\xi)) < W_\xi(\bar{p},\bar{q}) \cdot \bar{\bar{z}}_i + \varepsilon_\xi(\bar{p},\bar{q})$. From the Local Nonsatiation **LNS**, there exists $x_i^n(\xi) \to \bar{x}_i(\xi)$ such that $x_i^n := (x_i^n(\xi), \bar{x}_i(-\xi)) \subset P_i^r(\bar{x})$ for all n. Then, it is possible to choose n large enough so that $(x_i^n, \bar{\bar{z}}_i) \in B_i^{r\varepsilon}(\bar{p},\bar{q})$, which together with $x_i^n \in P_i^r(\bar{x})$ contradicts the fact that $B_i^{r\varepsilon}(\bar{p},\bar{q}) \cap \big(P_i^r(\bar{x}) \times Z_i^r\big) = \emptyset$ (by Claim 3.6). This ends the proof of (3.11).

Summing up over i the equalities (3.11), we get $\varepsilon(\bar{p},\bar{q}) = 0$, using the facts that $\sum_{i\in I}\bar{\bar{z}}_i = 0$ (Claim 3.4) and $\sum_{i\in I}\bar{x}_i = \sum_{i\in I} e_i$ (Claim 3.5). ■

Claims 3.4 - 3.7 shows that $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$.

3.7 The point $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$

Claim 3.8. *The list* $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.

Proof. Market clearing condition holds from claim **??** and claim **??**, and we have to prove that

$$(P_i(\bar{x}) \times Z_i) \cap B_i(\bar{p}, \bar{q}) = \emptyset, \text{ for every } i \in I.$$

Assume that it is not true, then for some $i \in I$, t and there exist $(x'_i, z'_i) \in (P_i(\bar{x}) \times Z_i) \cap B_i(\bar{p}, \bar{q})$. Therefore $\bar{p} \,\square\, (x'_i - e_i) \leq W(\bar{p}, \bar{q}) z'_i$. Since $\bar{x}$ is an attainable allocation and $\bar{\bar{z}} \in K_{\underline{v}}$, the definition of r implies that $\bar{x}_i \in \hat{X} \subset int\ B_L(0, r)$ *and* $\bar{\bar{z}}_i \in \hat{Z} \subset int\ B_L(0, r)$. Thus for t small enough, $(\bar{x}_i + t(x'_i - \bar{x}_i), \bar{\bar{z}}_i + t(z'_i - \bar{\bar{z}}_i)) \in (P^r_i(\bar{x}) \times Z_i) \cap B^r_i(\bar{p}, \bar{q})$. Therefore $(P^r_i(\bar{x}) \times Z_i) \cap B^r_i(\bar{p}, \bar{q}) \neq \emptyset$, which contradicts $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}^r, \mathcal{F}^r)$.

Hence $(\bar{x}, \bar{\bar{z}}, \bar{p}, \bar{q})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$. $\square$

4 Appendix

4.1 Proof of claims

Proof of Claim 1

Proof. $Q_\xi(p)$ is a cone can be seen easily. Let's say $Q_\xi(p)$ is not convex, then there exist $q_1(\xi) \in Q_\xi(p), q_2(\xi) \in Q_\xi(p)$ and $\mu q_1(\xi) + (1-\mu) q_2(\xi) \notin Q_\xi(p)$ for some $\mu \in (0,1)$. Then there exists $z_i(\xi)$ such that either $\{-\big(\mu q_1(\xi) + (1-\mu) q_2(\xi)\big) \cdot z_i(\xi) > 0$ and $V(\xi, p) z_i(\xi) \geq 0\}$ or $\{-\big(\mu q_1(\xi) + (1-\mu) q_2(\xi)\big) \cdot z_i(\xi) \geq 0$ and $V(\xi, p) z_i(\xi) > 0\}$. In first case, we conclude either $-q_1(\xi) \cdot z_i(\xi) > 0$ or $-q_2(\xi) \cdot z_i(\xi) > 0$ which together with $V(\xi, p) z_i(\xi) \geq 0$ contradict the fact that $q_1(\xi) \in Q_\xi(p), q_2(\xi) \in Q_\xi(p)$. In second case, we conclude that either $-q_1(\xi) \cdot z_i(\xi) \geq 0$ or $-q_2(\xi) \cdot z_i(\xi) \geq 0$ which together with $V(\xi, p) z_i(\xi) > 0$ contradict the fact that $q_1(\xi) \in Q_\xi(p), q_2(\xi) \in Q_\xi(p)$. Hence $Q_\xi(p)$ is convex. ∎

Proof of Claim 2

Proof. We will prove Claim 2 by the principal of mathematical induction. We will consider a $(T+1)$-date economy. Firstly, we will consider the financial structure $\mathcal{F}^{T-1}$, where assets are issued only at time $t = T-1$ and then show that $Q_{\mathcal{F}^T} = \prod_{\xi(t) = T-1} Q_\xi$. Then we consider another financial structure $\mathcal{F}^{t'}$, where assets are issued at time $t = \{t', t'+1, \cdots, T-1\}$ for some $0 < t' < T-1$, and assume that $Q_{\mathcal{F}^{t'}} = \prod_{\xi(t) \geq t'} Q_\xi$. Then, we will show for the financial structure $\mathcal{F}^{t'-1}$ (where assets are issued at time $t = \{t'-1, t', \cdots, T-1\}$), $Q_{\mathcal{F}^{t'-1}} = \prod_{\xi(t) \geq t'-1} Q_\xi$.

For the financial structure $\mathcal{F}^{T-1}$, where assets are issued only at time $t = T-1$, $Q_{\mathcal{F}^T} = \prod_{\xi(t) = T-1} Q_\xi$ is obvious. Now assuming $Q_{\mathcal{F}^{t'}} = \prod_{\xi(t) \geq t'} Q_\xi$ for the financial structure $\mathcal{F}^{t'}$, we will find $Q_{\mathcal{F}^{t'-1}}$ for the financial structure $\mathcal{F}^{t'-1}$.

We will independently consider the financial structure originating at ξ for all $\xi \in \mathcal{D}_{t'-1}$. Now we know that if we consider the financial structure $\mathcal{F}_\xi$ starting at node ξ, with assets at all the nodes $\xi' \in \mathcal{D}^+(\xi)$, then $Q_{\mathcal{F}_\xi} = \prod_{\xi' \in \mathcal{D}^+(\xi)} Q_{\xi'}$ (from induction assumption). Now $q \in Q_{\mathcal{F}_\xi}$, from the characterization of arbitrage-free prices, there exists $\mu \in \mathbb{R}^{\mathcal{D}^+(\xi)}$ such that $\mu^t W_\xi(q) = 0$, or,

$$\mu(\zeta^j) q^j = \sum_{\zeta > \zeta^j} \mu(\zeta) W_\zeta^j, \qquad \forall j \in \mathcal{J}.$$

Now we also issue certain financial assets at node ξ, and call this new financial structure $\mathcal{F}'_\xi$. Now, we take any $q \in Q_{\mathcal{F}_\xi}$, and find the corresponding $\mu \in \mathbb{R}^{\mathcal{D}^+(\xi)}$, and take $\mu(\xi) = 1$, then we will get $q'_0 = (q_\xi, q) \in Q_{\mathcal{F}'_\xi}$ corresponding to this μ. Also if we take $\mu(\xi) = \alpha$, then we will get $q'_1 = (\alpha q_\xi, q) \in Q_{\mathcal{F}'_\xi}$. Furthermore, we know that $Q_{\mathcal{F}'_\xi}$ is a cone, therefore $q'_2 = (q_\xi, \beta q) \in Q_{\mathcal{F}'_\xi}$. Therefore, $q' = (\alpha q_\xi, \beta q) \in Q_{\mathcal{F}'_\xi}$ for all $\alpha > 0, \beta > 0$. Therefore $Q_{\mathcal{F}'_\xi} = Q_\xi \times Q_{\mathcal{F}_\xi} = \prod_{\xi' \in \{\xi \cup \mathcal{D}^+(\xi)\}} Q_{\xi'}$

And when we combine all $\xi \in \mathcal{D}_{t'-1}$, we get $Q_{\mathcal{F}^{t'-1}} = \prod_{\xi(t) \geq t'-1} Q_\xi$. ■

Proof of Claim 3

Proof of Claim 3 follows from the definition of Π, and is obvious.

Proof of Claim 4

Proof. Assume K_v is not bounded. Then there exist a sequence $(p^n)_n \subset B_L(0,1)$ and a sequence $((z_i^n(\xi))_{i \in I, \xi \in (\mathcal{D} \setminus \mathcal{D}_T)}) \subset \prod_{i \in I, \xi \in (\mathcal{D} \setminus \mathcal{D}_T)} Z_i(\xi)$ such that for each n, and for every i, $\sum_{\xi' \in \mathcal{D}^-(\xi)} V_\xi(\xi', p) z_i(\xi') - q(\xi) \cdot z_i(\xi) \geq v_i^\xi, \quad \forall \xi \in \mathcal{D} \setminus \{\mathcal{D}_T \cup \{0\}\}$ and $\sum_{\xi' \in \mathcal{D}^-(\xi)} V_\xi(\xi', p) z_i(\xi') \geq v_i^\xi$, $\forall \xi \in \mathcal{D}_T$, and $-\sum_{i \in I} z_i(\xi) \in \sum_{i \in I} (\mathbf{A} Z_i(\xi) \cap \{V(\xi, p) \geq 0\})$ for all $\xi \in \mathcal{D} \setminus \mathcal{D}_T$, and $\sum_{i \in I} \sum_{\xi' \in \mathcal{D} \setminus \mathcal{D}_T} ||z_i^n(\xi')|| \to \infty$. If needed, we can assume that the sequence $(p^n)_n$ converges to $p \in B_L(0,1)$ by moving to a subsequence.

Without loss of generality, we can assume that there exist a node $\xi \in \mathcal{D}$ such that $\sum_{i \in I} \sum_{\xi' \in \mathcal{D}^+(\xi^-)} ||z_i^n(\xi')|| < \infty$. Then, we have $\sum_{i \in I} \sum_{\xi' \in \mathcal{D}^-(\xi)} ||z_i^n(\xi')|| \to \infty$. For each i, for each $\xi'' \in \mathcal{D}^-(\xi)$, the sequence $(z_i^n(\xi'') / \sum_{k \in I} \sum_{\xi' \in \mathcal{D}^-(\xi)} ||z_k^n(\xi')||$ is bounded hence we can assume that it converges to $\chi_i(\xi'')$. The vector $\chi_i(\xi'')$ belongs to $\mathbf{A} Z_i(\xi'')$ since $z_i^n(\xi'') \in Z_i(\xi'')$ for every n. Since, we have $\sum_{\xi' \in \mathcal{D}^-(\xi)} V_\xi(\xi', p) z_i(\xi') - q(\xi) \cdot z_i(\xi) \geq v_i^\xi$ and therefore $\sum_{\xi' \in \mathcal{D}^-(\xi)} V_\xi(\xi', p) \chi_i(\xi') \geq 0$ for every i. This will imply $\sum_{i \in I} \sum_{\xi' \in \mathcal{D}^-(\xi)} V_\xi(\xi', p) \chi_i(\xi') \geq 0$, and $\sum_{\xi' \in \mathcal{D}^-(\xi)} V_\xi(\xi', p) (\sum_{i \in I} \chi_i(\xi')) \geq 0$.

And $-\sum_{i \in I} z_i^n(\xi) \in \sum_{i \in I} (\mathbf{A} Z_i(\xi) \cap \{V(\xi, p) \geq 0\})$, implies $-\sum_{i \in I} \chi_i(\xi'') \in \mathbf{A}_{\xi''}(\mathcal{F})$ and also $\sum_{i \in I} \chi_i(\xi'') \in \mathbf{A}_{\xi''}(\mathcal{F})$. Therefore, $\sum_{i \in I} \chi_i(\xi'') = \{0\}$. Hence $\chi_i(\xi'') \in \mathbf{A}_{\mathcal{F}}(\xi'') \mathbf{A}_{\xi''}(\mathcal{F})$ for each i, and $\sum_{i \in I} \chi_i(\xi'') = \{0\}$ together implies $\chi_i(\xi'') = 0$ for every i. And therefore we get the following contradiction.

$$1 = \sum_{k \in I} \sum_{\xi'' \in \mathcal{D}^-(\xi)} (||z_k^n(\xi'')|| / \sum_{i \in I} \sum_{\xi' \in \mathcal{D}^-(\xi)} ||z_i^n(\xi')||) = \sum_{k \in I} \sum_{\xi'' \in \mathcal{D}^-(\xi)} ||\chi_k(\xi'')|| = 0.$$

Hence, K_v is bounded. ■

4.2 An economy with no equilibria

This example is taken from Magill and Quinzii [1996b]. In this example, the assets can be retraded. Later, we give an equivalent economy (equivalence between a retradable economy and non-retradable economy is shown in Angeloni and Cornet [2006]) with a financial structure, where no assets can be retraded. In this example, we consider a three-date economy with the event-tree $\mathcal{D}$ is represented by:

$$\mathcal{D} = \{\xi_0, (\xi_1, \xi_2), (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22})\}, \tag{4.1}$$

and the two long-lived assets (numéraire assets) issued with dividend processes

$$V^1 = (0, (0, 0), (1, 0, 1, 0)), \text{ and } V^2 = (0, (0, 0), (0, 1, 0, 1)). \tag{4.2}$$

Two agents 1 and 2 with their respective initial endowments and utility functions:

$$\begin{aligned} \omega^1 &= (0, (1+\varepsilon, 1-\varepsilon), (1, 1, 1, 1)), \\ \omega^2 &= (0, (1-\varepsilon, 1+\varepsilon), (1, 1, 1, 1)), \end{aligned} \tag{4.3}$$

$$\begin{aligned} u^1(x) &= x_1^\alpha + x_2^\alpha + x_1^\beta x_{11}^\alpha + x_{12}^\alpha + x_2^\beta x_{21}^\alpha + x_{22}^\alpha, \\ u^2(x) &= x_1^{\alpha+\beta} + x_2^{\alpha+\beta} + x_{11}^\alpha + x_{12}^\alpha + x_{21}^\alpha + x_{22}^\alpha, \end{aligned} \tag{4.4}$$

with

$$\alpha > 0, \beta > 0, \alpha + \beta < 1, 0 < |\varepsilon| < \varepsilon^* \text{ for some very small } \varepsilon^*.$$

Claim 4.1. *The economy defined by (4.1) to (4.4) doesn't have any equilibrium.*

Proof. The payoff matrix at date 1 is:

$$\pi \ o \ \xi_0^+ = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} q_1(\xi_1) & q_2(\xi_1) \\ q_1(\xi_2) & q_2(\xi_2) \end{bmatrix}.$$

The rank of the matrix $\pi \ o \ \xi_0^+$ can be either 1 or 2.
Case **(i)**: rank $(\pi \ o \ \xi_0^+) = 2$. For $\varepsilon = 0$, the initial endowment is Pareto optimal since

$$\triangledown u^1(\omega^1) = \triangledown u^2(\omega^2) = ((\alpha+\beta, \alpha+\beta), (\alpha, \alpha, \alpha, \alpha))$$

(where the date 0 component of the gradient is omitted). Thus there is a unique contingent market equilibrium given by

$$\pi = (1, (\alpha+\beta, \alpha+\beta), (\alpha, \alpha, \alpha, \alpha))$$
$$x^1 = x^2 = (0, (1, 1), (1, 1, 1, 1)). \tag{4.5}$$

From the analysis of contingent market equilibrium of an economy parametrized by the endowments that if the endowments stay close to Pareto optimal endowments then the equilibrium remains unique. Thus there exist $\varepsilon^* > 0$ such that there is a unique contingent market equilibrium for all ε such that $0 < |\varepsilon| < \varepsilon^*$. For all such ε the contingent market equilibrium is given by (4.5), since the budget constraint of both agents are still satisfied. Substituting the price vector given in (4.5) into the payoff matrix, we get rank $(\pi \ o \ \xi_0^+) = 1$ (contradiction).

Case **(ii)**: rank $(\pi \ o \ \xi_0^+) = 1$. In this case, $\frac{\pi_{11}}{\pi_{21}} = \frac{\pi_{12}}{\pi_{22}}$. Hence, one asset issued at date 0 is redundant. And since the payoff of other asset is positive in both states at date 1. Since marginal utility of consumption at date 0 is zero (as there is no consumption at date 0), both agents are willing to buy this asset at date 0, allowing no trade at date 0. Therefore, there is no income transfer between nodes ξ_1 and ξ_2. Since the utility functions are separable between consumption in the subtrees $\mathcal{D}(\xi_1)$ and $\mathcal{D}(\xi_2)$, an equilibrium (x, π) must be the equilibrium for the economy on the subtree $\mathcal{D}(\xi_j)$ beginning at node ξ_j for each $j = \{1, 2\}$.

For the $\mathcal{D}(\xi_1)$ economy, the utility function and endowments are given by

$$v^1(x_1, (x_{11}, x_{12})) = x_1^\alpha + x_1^\beta x_{11}^\alpha + x_{12}^\alpha, \quad (\omega_1^1, (\omega_{11}^1, \omega_{12}^1)) = (1+\varepsilon, (1, 1))$$
$$v^2(x_1, (x_{11}, x_{12})) = x_1^{\alpha+\beta} + x_{11}^\alpha + x_{12}^\alpha, \quad (\omega_1^2, (\omega_{11}^2, \omega_{12}^2)) = (1-\varepsilon, (1, 1)).$$

And for the $\mathcal{D}(\xi_2)$ economy,

$$v^1(x_2, (x_{21}, x_{22})) = x_2^\alpha + x_2^\beta x_{21}^\alpha + x_{22}^\alpha, \quad (\omega_2^1, (\omega_{21}^1, \omega_{22}^1)) = (1-\varepsilon, (1, 1))$$
$$v^2(x_2, (x_{21}, x_{22})) = x_2^{\alpha+\beta} + x_{21}^\alpha + x_{22}^\alpha, \quad (\omega_2^2, (\omega_{21}^2, \omega_{22}^2)) = (1+\varepsilon, (1, 1)).$$

The only difference between these two economies is that in the $\mathcal{D}(\xi_1)$ economy agent 1 is richer than agent 2 in terms of initial endowment, and conversely in $\mathcal{D}(\xi_2)$ economy. Let ν_j^1, ν_j^2 denote the marginal utilities of income of agents 1 and 2 in the $\mathcal{D}(\xi_j)$ economy. Then solving first order conditions for equilibrium consumption bundles and market clearing condition, we show the following:

(i) $\nu_j^1 < \nu_j^2 \Rightarrow x_j > 1, x_{j1}^1 > 1, x_{j2}^1 > 1 \Rightarrow \pi_{j1}/\pi_{j2} > 1$,
(ii) $\nu_j^1 = \nu_j^2 \Rightarrow x_j = 1, x_{j1}^1 = 1, x_{j2}^1 = 1 \Rightarrow \pi_{j1}/\pi_{j2} > 1$,
(iii) $\nu_j^1 > \nu_j^2 \Rightarrow x_j < 1, x_{j1}^1 < 1, x_{j2}^1 < 1 \Rightarrow \pi_{j1}/\pi_{j2} > 1$.

If $\varepsilon > 0$, then in the $\mathcal{D}(\xi_1)$ equilibrium, agent 1 is richer than agent 2 so that **(i)** must occur; in the $\mathcal{D}(\xi_2)$ equilibrium , agent 1 is richer than agent 2 so that **(iii)** must occur, and conversely if $\varepsilon < 0$. Thus, if $\varepsilon > 0$, then $\pi_{11}/\pi_{12} > 1$ and $\pi_{21}/\pi_{22} < 1$; and converse when $\varepsilon < 0$. It follows that rank$(\pi \ o \ \xi_0^+) = 2$ (contradiction).

Hence, there is no equilibria for the economy defined by (4.1) to (4.4).

■

Equivalent non-retradable economy

When we rewrite the total payoff matrix using the notation used in our model (taken from Angeloni and Cornet [2006]), then the total payoff matrix is:

$$W_{\mathcal{F}}(q) := \begin{bmatrix} -q^1 & -q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^3 & q^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^5 & q^6 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Claim 4.2. *The set $Q_{\mathcal{F}}$ is not convex.*

Proof. We know that $q \in Q_{\mathcal{F}}$ if and only if there exist $\mu = (1, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) \in \mathbb{R}^7_{++}$ such that $\mu^T W_{\mathcal{F}}(q(\mu)) = 0$ (by the characteristion theorem of arbitrage-free prices), or

$$q_1 = \mu_3 + \mu_5,\ q_2 = \mu_4 + \mu_6,\ q_3 = \mu_3/\mu_1,\ q_4 = \mu_4/\mu_1,\ q_5 = \mu_5/\mu_2,\ \text{and } q_6 = \mu_6/\mu_2. \quad (4.6)$$

We can see that for $q_1 = (6, 11, 3, 5, 3, 6)$, there exist $\mu = (1, 1, 1, 3, 5, 3, 6) \in \mathbb{R}^7_{++}$ such that Equation (4.6) is satisfied. Also for $q_2 = (9, 20, 3, 7, 3, 6)$, there exist $\mu_2 = (1, 2, 1, 6, 14, 3, 6) \in \mathbb{R}^7_{++}$ such that Equation (4.6) is satisfied. But for $q = \frac{1}{2}q_1 + \frac{1}{2}q_2 = (7.5, 15.5, 3, 6, 3, 6)$, there does not exist any $\mu \in \mathbb{R}^7_{++}$ such that Equation (4.6) is satisfied. Hence, $Q_{\mathcal{F}}$ is not convex.

Suppose for $q = (7.5, 15.5, 3, 6, 3, 6)$, there exist a $\mu = (1, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) \in \mathbb{R}^7_{++}$ such that Equation (4.6) is satisfied. Then, we have $\frac{q_3}{q_4} = \frac{\mu_3}{\mu_4} = \frac{1}{2} = \frac{q_5}{q_6} = \frac{\mu_5}{\mu_6}$, therefore $\frac{7.5}{15.5} = \frac{q_1}{q_2} = \frac{\mu_3+\mu_5}{\mu_4+\mu_6} = \frac{1}{2}$ (a contradiction). ■

Result of this paper doesn't conclude any inference for the economy defined above as the set of arbitrage-free prices is not convex in this case.

References

L. Angeloni and B. Cornet. Existence of financial equilibria in a multi-period stochastic economy. *Advances in Mathematical Economics*, 8:1–31, 2006.

Z. Aouani and B. Cornet. Existence of financial equilibria with restricted participation. *Journal of Mathematical Economics*, 45:772–786, 2009.

A. Araujo, M. Pascoa, and J. Orrillo. Incomplete markets, continuum of states and default. *Economic Theory*, 11:205–213, 1997.

T. C. Bergstrom. How to discard 'free disposability' - at no cost. *Jouornal of Mathematical economics*, 3:131–134, 1976.

P. Bich and B. Cornet. Existence of financial equilibria: space of transfer of fixed dimension. Working Paper, University of Paris 1, 1997.

J. M. Bonnisseau and O. Lachiri. Dréze's criterion in a multi-period economy with stock markets. *Jouornal of Mathematical economics*, 40:493–513, 2004.

D. Cass. Competitive equilibrium with incomplete markets. *CARESS Working Paper No. 84-09, University of Pennsylvania*, 1984.

D. Cass, P. Siconolfi, and A. Villanacci. Generic regularity of competitve equilibria with restricted participation. *Jouornal of Mathematical economics*, 36:61–76, 2001.

A. Citanna and K. Schmedders. Excess price volatility and price innovation. *Economic Theory*, 26:559–587, 2005.

A. Mas Colell and W. R. Zame. A new proof of existence of equilibrium in incomplete markets economies. *Journal of Mathematical Economics*, 26:85–101, 1996.

A. Mas Colell, M. D. Whinston, and J. R. Green. *Microeconomic theory*. Oxford University Press, New York, 1995.

B. Cornet and A. Ranjan. A remark on the set of arbitrage-free prices in a multi-period model. Working Paper, University of Paris 1, 2012b.

G. Debreu. *Theory of Value*. Yale University Press, 1959.

D. Duffie. Stochastic equilibria with incomplete financial markets. *Journal of Economic Theory*, 41:404–416, 1987.

D. Duffie and W. Shafer. Equilibrium in incomplete markets I: Basic model of generic existence. *Journal of Mathematical Economics*, 14:285–300, 1985.

D. Duffie and W. Shafer. Equilibrium in incomplete markets II: Generic existence in stochastic economies. *Journal of Mathematical Economics*, 15:199–216, 1986.

M. Florenzano and P. Gourdel. T-period economies with incomplete markets. *Economic Letters*, 44:91–97, 1994.

M. Florenzano, P. Gourdel, and M. Pascoa. Overlapping generation models with incomplete markets. *Journal of Mathematical Economics*, 36:201–218, 2001.

D. Gale and A. Mas-Colell. An equilibrium existence theorem for a general model without ordered preferences. *Journal of Mathematical Economics*, 2:9–15, 1975.

J. Geanakoplos and H. Polemarchakis. *Uncertainty, information and communication: Essays in honor of Kenneth J. Arrow*, volume III, chapter Existence, regularity and constrainted suboptimality of competitive allocations when the asset market is incomplete, pages 65–95. Heller et al. Ed. Cambridge University Press, Cambridge, 1986.

D. K. Levine and W. R. Zame. Debt constraint and equilibrium in infinite horizon economies with incomplete markets. *Journal of Mathematical Economics*, 26:103–131, 1996.

M. Magill and M. Quinzii. Infinite horizon incomplete markets. *Econometrica*, 62:853–880, 1994.

M. Magill and M. Quinzii. ncomplete markets over an infinite horizon: Long-lived securities and speculative bubbles. *Journal of Mathematical Economics*, 26:133–170, 1996a.

M. Magill and M. Quinzii. *Theory of Incomplete Markets*. Cambridge: MIT University Press, 1996b.

M. Magill and W. Shafer. Incomplete markets. In North-Holland Amsterdam, editor, *Handbook of Mathematical Economics*, volume 4. W. Hildenbrand and H. Sonnenschein, eds, 1991.

Filipe Martins-da-Rocha and Leila Triki. Equilibria in exchange economies with financial constraints: Beyond the Cass-trick. Working Paper, University of Paris 1, 2005.

P. K. Monteiro. Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles. *Journal of Mathematical Economics*, 26:133–170, 1996.

P. K. Monteiro and M. Pascoa. Discreteness of equilibria in incomplete markets with a continuum of states. *Journal of Mathematical Economics*, 33:229–237, 2000.

J. Orrillo. Default and exogenous collateral in incomplete markets with a continuum of states. *Journal of Mathematical Economics*, 35:151–165, 2001.

P. Siconolfi. Equilibrium with asymmetric constraints on portfolio holdings and incomplete financial markets. In *M. Galeotti, L. Geronazzo and F. Gori, eds, Non-Linear Dynamics in Economics and Social Sciences, Societa' Pitagora*, pages 271–292, 1989.

J. Werner. Equilibrium in economies with incomplete financial markets. *Journal of Economic Theory*, 36:110–119, 1985.

J. Werner. Equilibrium with incomplete markets without ordered preferences. *Journal of Economic Theory*, 49:379–382, 1989.